Thermodynamics of Ideal Gas in Cosmology

Ying-Qiu Gu[*]

Department of Physics, Fudan University, Shanghai 200433, China

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The equation of state and the state functions for the gravitational source are necessary conditions for solving cosmological model and stellar structure. The usual treatments are directly based on the laws of thermodynamics, and the physical meanings of some concepts are obscure. This letter show that, we can actually derive all explicit fundamental state functions for the ideal gas in the context of cosmology via rigorous dynamical and statistical calculation. These relations have clear physical meanings, and are valid in both non-relativistic and ultra-relativistic cases. Some features of the equation of state are important for a stable structure of a star with huge mass.

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The equation of state and the relations between thermodynamic variables are necessary conditions for solving cosmological model and stellar structure. Obviously to get the realistic state functions for all kinds of matter is a difficult problem due to the lack of the knowledge for some materials such as the dark matter. However, for the most important gravitational source in general relativity, i.e., the ideal gases or the perfect fluid model, we can rigorously solve the problem. Some ordinary but important features of the EOS may be valid for all kinds of matter.

The cooling mechanism of the expanding universe is one of the most interesting problem of the students. The answers given by the standard cosmological text books [1] and the pedagogical articles [2, 3, 4] are usually $T \propto a^{-1}$, which is derived from the classical thermodynamics. In [5, 6, 7, 8, 9], the author solved the thermodynamical relations according to the Gibbs' law

$$TdS = d(\rho V) + PdV. \tag{1}$$

The problem looks underdetermined due to (1) including a number of undetermined quantities. So we have to introduce some new concepts and relations such as multi fluid with energy exchange, the apparent horizon entropy, the decay of vacuum, to make the equations closed.

As pointed out in [10], these approaches seem not convincing the students and satisfying their

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curiosity. Dividing the velocity of particles into Hubble velocity $v_h = ra'(t)$ and peculiar velocity $v_{pec} = ar'(t)$, and then analyzing the geodesic of a particle, in [10] the author qualitatively reached the conclusion $p \propto a^{-1}$. Then by mass-energy relation $E^2 = p^2 + m^2$, he concluded that the cosmic temperature should be $T \propto a^{-1}$ for the ultra-relativistic gas but $T \propto a^{-2}$ for the non-relativistic gas. In what follows, we rigorously derive this relation $T = T(a)$ and some other state functions of the ideal gases.

In the Friedman-Robertson-Walker(FRW) space-time, we have the line element

$$ds^2 = dt^2 - a^2(t) \left(dr^2 + f^2(r)d\theta^2 + f^2(r)\sin^2\theta d\varphi^2 \right), \tag{2}$$

where

$$f = \begin{cases} \sin r & \text{if} \quad \kappa = 1, \\ r & \text{if} \quad \kappa = 0, \\ \sinh r & \text{if} \quad \kappa = -1. \end{cases} \tag{3}$$

The evolution equation is given by the Friedman equation

$$\dot{a}^2 + \kappa - \frac{1}{3}\Lambda a^2 = \frac{8\pi G}{3}\rho a^2, \tag{4}$$

where the over dot stands for $\frac{d}{dt}$. The energy-momentum tensor of the perfect fluid and radiation

$$T^{\mu\nu} = (\rho + P)U^\mu U^\nu - Pg^{\mu\nu} \tag{5}$$

satisfies the conservation law

$$\frac{d(\rho a^3)}{da} = -3Pa^2. \tag{6}$$

For a given specific equation of state $P = P(\rho)$, we can solve the function $\rho = \rho(a)$, e.g. for radiation $P = \frac{1}{3}\rho$, then we have $\rho = \alpha a^{-4}$. Adding initial values $a(t_0)$, (4) forms a problem of determining the solution. The problem is then thoroughly solved in theoretical sense.

In the microscopic view, for the ideal gases and photons, the particles are only driven by average gravity and move along geodesics, and the collisions between particles can be treated as instantaneous behavior. So all thermodynamic functions can be rigorously solved according to dynamics and statistics. For convenience, we discuss the thermodynamics of the ideal gases in the context of cosmology. If we take (4) and (5) as the macroscopic definitions of ρ and P, and comparing it with the expectation value of the microscopic energy-momentum tensor[1, 11], we can inversely derive the equation of state $P = P(\rho)$ and some other relations. The final results are

independent of any concrete space-time, so they are actually valid for the ideal gas in any curved space-time due to the principle of equivalence. This is the basic idea of this study.

At first, we discuss the Gibbs' law (1) in context of cosmology. We have the following conclusion.

Theorem 1 *For the materials with energy-momentum tensor (5), the cosmological principle or the FRW metric implies the isentropic process* $dS \equiv 0$.

This can be checked as follows. Let $V = \Omega a^3$, where Ω is any given constant comoving volume independent of a, substituting it into (1) and using energy-momentum conservation law (6), we find $dS \equiv 0$.

To analyze the dynamics of the particles, we introduce a useful lemma

Lemma 2 *If the line element of the orthogonal subspace has the following form,*

$$ds^2 = \boldsymbol{A}(t)dt^2 + \widetilde{g}_{\mu\nu}(t)dx^\mu dx^\nu, \tag{7}$$

where $\boldsymbol{A}$ *and* $\widetilde{g}_{\mu\nu}$ *only depend on the coordinate* t, *then the geodesic in this subspace can be solved by*

$$\frac{dx^\mu}{ds} = \widetilde{g}^{\mu\nu}C_\nu, \quad \frac{dt}{ds} = \sqrt{\frac{1}{\boldsymbol{A}}(1 - \widetilde{g}^{\mu\nu}C_\mu C_\nu)}, \tag{8}$$

where C_μ *are constants, and* $\widetilde{g}^{\mu\nu}\widetilde{g}_{\nu\alpha} = \delta^\mu_\alpha$.

Lemma 2 can be checked directly. For the FRW metric (2), the line element in the orthogonal subspace (t, r) is given by $ds^2 = dt^2 - a(t)^2 dr^2$. According to Lemma 2, we have the geodesic equation

$$\frac{d}{ds}r = \frac{C}{a^2}, \quad \frac{d}{ds}t = \frac{1}{a}\sqrt{a^2 + C^2}, \tag{9}$$

where C is a constant only depends on the initial data. By (9) we get the drifting speed of a particle

$$v \equiv \frac{adr}{dt} = \frac{C}{\sqrt{a^2 + C^2}}, \quad C = \frac{v_0}{\sqrt{1 - v_0^2}}a(t_0). \tag{10}$$

So the momentum of a particle $p = \frac{m_n v}{\sqrt{1-v^2}}$ satisfies

$$p(t)a(t) = p(t_0)a(t_0), \tag{11}$$

where m_n is the proper mass of the particle. For the massless photons, we can check that the wavelength $\lambda(t)$ satisfies $\frac{\lambda(t)}{a(t)} \equiv \frac{\lambda_0}{a_0}$, so their momentum p also satisfy (11). Although (11) is derived in subspacetime (t, r), but it is suitable for all particles due to the symmetry of the FRW metric.

The relation between momentum p and the kinetic energy K is given by

$$p^2 = K(K + 2m). \tag{12}$$

For fermions, we have the energy distribution of Fermi-Dirac statistics due to the Pauli exclusion principle. The Fermi-Dirac distribution achieved great successes in the condensed matter physics[12], where the influence of Pauli exclusion principle is important. However for the ideal gas, the situation is different, where one can hardly connect the drifting movement of the particles with Pauli principle. Secondly, the proper mass of a species of particles is a constant, which seems inadequate to become a part of random variable, namely the random variable should be kinetic energy K rather the total energy $E = K+m$. Accordingly, the statistical distribution of the kinetic energy K for ideal gas should be

$$\mathcal{F}(K)dK = \sqrt{\frac{4K}{\pi(kT)^3}} \exp\left(-\frac{K}{kT}\right) dK. \tag{13}$$

The nonrelativistic limit of (13) is just the Maxwell velocity distribution, in which the random variable is the velocity of a particle v. Considering the following reasons, (13) is a more general form. First, the partition function of any ensemble is related to the random energy of its components. Second, the Maxwell distribution is not Lorentz invariant, but (13) is suitable for both the non-relativistic and ultra-relativistic cases. Third, (13) is suitable for multi-particle case. At last, (13) is convenient for mathematical treatment. In the following calculation, we only need the first and second order moments rather than the concrete $\mathcal{F}(K)$ for the following discussion, so we assume

$$\bar{K} = \int_0^\infty K\mathcal{F}(K)dK = \frac{3}{2}kT, \qquad \int_0^\infty K^2\mathcal{F}(K)dK = \frac{3}{2\sigma}(kT)^2, \tag{14}$$

where σ is a constant depending on specific distribution function $\mathcal{F}(K)$, and for (13) we have $\sigma = \frac{2}{5}$. For the drifting movement of a particle with proper mass m_n, by (11) we have $p_n^2 = \frac{C_n}{a^2}$, where C_n are constants only depending on the initial data at $t = t_0$. Then on one hand, for all particles we have the mean square momentum directly

$$\bar{p}^2 = \frac{C_0}{a^2}, \tag{15}$$

where C_0 is a constant only determined by initial data at t_0.

One may argue that (15) is probably broken by the collision of the particles. The following Lemma shows that (15) holds in statistical sense.

Lemma 3 *The mean square momentum of the ideal gas is independent of the elastic collision of the particles.*

Proof For any elastic collision, we have momentum conservation law

$$\vec{p}_1 + \vec{p}_2 = \vec{P}_1 + \vec{P}_2, \tag{16}$$

and

$$p_1^2 + p_2^2 = P_1^2 + P_2^2 + 2(\vec{P}_1 \cdot \vec{P}_2 - \vec{p}_1 \cdot \vec{p}_2). \tag{17}$$

Taking average for (17), we have

$$\bar{p}^2 = \bar{P}^2 + \Delta. \tag{18}$$

Since the elastic collision is a reversible process, in statistical sense, we have the exactly equal numbers of reversible process of (16), so we also have

$$\bar{P}^2 = \bar{p}^2 + \Delta. \tag{19}$$

Comparing (18) with (19), we have $\Delta = 0$ and $\bar{p}^2 = \bar{P}^2$. The proof is finished.

Since collision is finished instantaneously, (15) holds for all time t.

On the other hand, according to statistical principle, by the moments (14) we have

$$
\begin{aligned}
\bar{p}^2 &= \sum_n \int_0^\infty \frac{N_n}{N} p_n^2 \mathcal{F}(K_n) dK_n \\
&= \sum_n \int_0^\infty \frac{N_n}{N} K_n(K_n + 2m_n) \mathcal{F}(K_n) dK_n \\
&= \sum_n kT \frac{N_n}{N}(\frac{3}{2\sigma}kT + 3m_n),
\end{aligned} \tag{20}
$$

where N_n is the number of particles with mass m_n in the volume $V = \Omega a^3$, and $N = \sum_n N_n$ is the total number of particles in the volume. Comparing (20) with (15), we get the equation of $T(a)$ as follows

$$kT\left(kT + 2\sigma\bar{m}\right) = \left(\frac{\sigma\bar{m}b}{a}\right)^2, \qquad a = \frac{\sigma\bar{m}b}{\sqrt{kT\left(kT + 2\sigma\bar{m}\right)}}, \tag{21}$$

where $\bar{m} = \sum_n \frac{N_n}{N}m_n$ is the average mass of all particles, and b is a constant only depending on initial data. Solving (21), we get

Theorem 4 *The cosmological temperature of universe with ideal gas satisfies*

$$kT = \frac{\sigma\bar{m}b^2}{a(a + \sqrt{a^2 + b^2})}, \tag{22}$$

where b is determined by the initial data a_0 and T_0

$$\frac{b}{a_0} = \sqrt{\frac{kT_0}{\sigma\bar{m}}\left(2 + \frac{kT_0}{\sigma\bar{m}}\right)}. \tag{23}$$

By the theorem we find the cosmic temperature is different from the results simply obtained from classical thermodynamics. In what follows we establish the relation between ρ and T, as well as the equation of state. For convenience of the following calculation, we make transformation $d\tau = a^{-1}dt$ to transform the coordinate system into the conformal form, then the line element (2) becomes

$$ds^2 = a^2(\tau)\left(d\tau^2 - dr^2 - f^2(r)d\theta^2 - f^2(r)\sin^2\theta d\varphi^2\right). \tag{24}$$

The Lagrangian of the cosmological model is given by[1, 13, 14]

$$\mathcal{L} = \frac{1}{16\pi G}(R - 2\Lambda) - \sum_k m_k\sqrt{1 - v_k^2}\delta^3(\vec{x} - \vec{X}_k), \tag{25}$$

where $\vec{X}_k$ is the coordinate of k-th particle, the scalar curvature

$$R = 6\frac{a'' + \kappa a}{a^3}, \tag{26}$$

in which the prime stands for $\frac{d}{d\tau} = a\frac{d}{dt}$, and the drifting speed of k-th particle

$$\vec{v}_k = (v_r, v_\theta, v_\varphi)_k = \left(\frac{dr}{d\tau}, \frac{f d\theta}{d\tau}, \frac{f\sin\theta d\varphi}{d\tau}\right)_k$$

are independent variables related to a for variation. Noticing that $\delta^3(\vec{x} - \vec{X}_k) \propto a^{-3}$, by variation of $I = \int \mathcal{L}a^4 dt d\Omega$ with respect to a, we get

$$a'' + \kappa a - \frac{2}{3}\Lambda a^3 = \frac{4\pi G}{3\Omega}\sum_{X_k\in\Omega} m_k\sqrt{1 - v_k^2}, \tag{27}$$

where Ω is any given comoving volume with volume element $d\Omega = f^2\sin\theta dr d\theta d\varphi$, which is independent of a. Substituting (10) into (27), we get

$$a'' + \kappa a - \frac{2}{3}\Lambda a^3 = \frac{4\pi G}{3\Omega}\sum_{X_k\in\Omega} \frac{m_k a}{\sqrt{a^2 + b_k^2}}. \tag{28}$$

Multiply (28) by a' and integrate it, again by (10) we have

$$\begin{aligned}
a'^2 + \kappa a^2 - \frac{1}{3}\Lambda a^4 &= \frac{8\pi G}{3\Omega}\sum_{X_k\in\Omega} \frac{m_k a}{\sqrt{1 - v_k^2}} + C_1, \\
&= \frac{8\pi G}{3\Omega}\sum_{X_k\in\Omega} (K_k + m_k)a + C_1,
\end{aligned} \tag{29}$$

where C_1 is a constant, for classical particles $C_1 = 0$. But for nonlinear spinors $C_1 < 0$, which resist the universe to become singular[14]. Making statistical average and using (22), we get

$$a'^2 + \kappa a^2 - \frac{1}{3}\Lambda a^4 = \frac{8\pi G}{3}\rho a^4 = \frac{8\pi G}{3}\bar{\rho}\left(1 + \frac{3}{2}\frac{\sigma b^2}{a(a + \sqrt{a^2 + b^2})}\right)a^4, \tag{30}$$

where ρ and $\bar{\rho}$ are respectively kinematic and static mass density defined by

$$\rho = \frac{1}{V} \sum_{X_k \in V} E_k = \frac{1}{V} \sum_{X_k \in V} (K_k + m_k), \quad \bar{\rho} = \frac{1}{V} \sum_{X_k \in V} m_k = \frac{\varrho}{a^3}, \tag{31}$$

in which ϱ is the comoving density independent of a. Comparing (30) with (4), we get

Theorem 5 *For the ideal gas in cosmology, the mass density satisfies*

$$\begin{aligned} \rho &= \bar{\rho}\left(1 + \frac{3}{2}\frac{\sigma b^2}{a(a + \sqrt{a^2 + b^2})}\right) = \bar{\rho}\left(1 + \frac{3}{2}\frac{kT}{\bar{m}}\right) = \bar{\rho}\left(1 + \frac{\bar{K}}{\bar{m}}\right), \\ &= \varrho_0[kT(kT + 2\sigma\bar{m})]^{\frac{3}{2}}\left(1 + \frac{3}{2}\frac{kT}{\bar{m}}\right), \end{aligned} \tag{32}$$

where ϱ_0 is a constant.

Substituting (32) into (6), we get

Theorem 6 *The equation of state for ideal gas in context of cosmology is given by*

$$P = \frac{\sigma\bar{\rho}b^2}{2a\sqrt{a^2 + b^2}} = \frac{NkT}{V}\left(1 - \frac{kT}{2(\sigma\bar{m} + kT)}\right), \tag{33}$$

$$= \frac{\varrho_0[kT(kT + 2\sigma\bar{m})]^{\frac{5}{2}}}{\bar{m}(kT + 2\sigma\bar{m})}\left(1 - \frac{kT}{2(\sigma\bar{m} + kT)}\right). \tag{34}$$

The final expressions of state functions (32) and (34) are independent of any parameters of the space-time, and the metric (24) is only used as 'piston-cylinder' to drive the particles in the derivation. So the state functions (32) and (34) are actually valid for all cases of ideal gases in general relativity. From the above calculation, we have the following conclusions.

1. In cosmology, the thermodynamic process is isentropic, it is always at the state of maximum entropy.

2. The temperature of the Universe varies as (22), it does not simply decrease in proportion to the inverse of scale factor.

3. The equation of state (34) slightly differs from the usual result of thermodynamics. By (32) and (34), we find the EOS $P = P(\rho)$ satisfying the following increasing and causal conditions

$$0 \leq C_s \leq \frac{1}{3}, \qquad C_s'(\rho) \geq 0, \qquad P \to \begin{cases} P_0\rho^\gamma, & (\gamma > 1, T \to 0), \\ \frac{1}{3}\rho, & (T \to \infty), \end{cases} \tag{35}$$

in which $C_s = \sqrt{P'(\rho)}$ is the velocity of sound in the materials, so (32) and (34) are reasonable in physics.

The state functions (32) and (34) can be derived in different ways[11], and such fluid can form stable stars with any given large gravitational mass. (35) is a quite common requirement for the EOS of gravitational source. More detailed analysis shows that, for any materials with the EOS satisfies the conditions (35), they can always form static and stable star. This conclusion can be easily checked as follows.

For the static spherically symmetrical space-time

$$ds^2 = B(r)dt^{\ 2} - A(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{36}$$

the exact solutions can be formally solved[15, 16, 17, 18]. In these solutions we usually expressed the mass density ρ and pressure P as the functions of (A, B) and their derivatives. Theoretically, such solutions form the general solution to the problem. However, in such expressions the properties of the EOS is obscure, and then most of the solutions are unrealistic in physics.

The realistic static asymptotically flat space-time with spherical symmetry can be converted into the following initial problem of an ordinary differential equation system, which can be easily solved numerically. If the EOS of the gravitational source satisfies (35), the dynamics for the space-time is equivalent to the following initial problem[1, 19],

$$\begin{align}
M'(r) &= 4\pi G\rho r^2, \qquad M(0) = 0, \tag{37}\\
\rho'(r) &= -\frac{(\rho + P)(4\pi GPr^3 + M)}{C_s^2(r - 2M)r}, \qquad \rho(0) = \rho_0, \tag{38}
\end{align}$$

where $M(r)$ is the ADM mass within the ball of radial coordinate r. For any given $\rho_0 > 0$ we get a unique solution with finite radius $R < \infty$. The metric components are then calculated by

$$A = \left(1 - \frac{2M}{r}\right)^{-1}, \qquad B = \exp\left(-\int_r^R \frac{2(4\pi GPr^3 + M)}{r(r - 2M)}dr\right). \tag{39}$$

In the region $r \geq R$, we have $\rho(r) = P(r) = 0$. Similar to the case calculated in [19], all solutions of (37) and (38) are singularity-free. That is, we always have $0 \leq \rho \leq \rho_0$ and $R_s = 2M(R) < R$.

The condition $\gamma > 1$ is necessary and interesting, which leads to the finite radius of the star $R < \infty$ due to $C_s(\rho) \to 0, (\rho \to 0)$ in (38). But in the case $\gamma = 1$, we get $R \to \infty$, and then the space-time is no longer asymptotically flat. This result reveals that, if EOS satisfies (35), for a star with any given large mass, we have regular solutions in equilibrium with finite ρ_0. The EOS of the matter is decisive for the fate of a star.

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